American Basket and Spread Option pricing by a Simple Binomial Tree

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Abstract

In this article we address the problem of valuing and hedging American options on baskets and spreads, i.e., on portfolios consisting of both long and short positions. We adopt the main ideas of the Generalized Lognormal (GLN) approach introduced in Borovkova et al. (2007) and extend them to the case of American options. We approximate the basket price process by a suitable Geometric Brownian motion, shifted by an arbitrary parameter and possibly reflected over the x-axis. These adjustments to the GBM are necessary for dealing with negative basket values and possible negative skewness of basket increments' distribution. We construct a simple binomial tree for an arbitrary basket, by matching the basket's volatility, and evaluate our approach by comparing the binomial tree option prices to those obtained by other methods, whenever possible. Moreover, we evaluate the delta-hedging performance of our method and show that it performs remarkably well, in terms of both option pricing and delta hedging. The main advantages of our method is that it is simple, computationally extremely fast and efficient, while providing accurate option prices and deltas.
Keywords: American basket options, Geometric Brownian Motion, GLN approach, binomial tree.

1. Introduction

A *basket option* is an option whose underlying asset is a basket, i.e., a portfolio of assets. Closely related to it is a *spread option*, where the underlying value is the spread, i.e., the difference in prices of two or more assets. Basket and, particularly, spread options are very popular in commodity markets, where producers are often exposed to spreads between a raw material (e.g., crude oil, fuel or soybean) and end products (e.g., gasoline and heating oil, electricity, or soybean oil and soybean meal). Basket and spread options are traded over-the-counter as well as on exchanges, and often have an American exercise feature.

It is well-known that there is no close form solution for the price of an American option, even when it is written on a single asset following Geometric Brownian motion. The most common method of valuing American options is the binomial tree method, introduced by Cox and Rubinstein (1979). Another way to value an American option is by Monte Carlo simulations, as suggested by Longstaff and Schwartz (2001). However, when the underlying value of an American option depends on several assets, such as in the case of basket or spread options, both binomial and simulation method quickly become either too slow or completely untractable. For example, to value an American basket option by a binomial tree method, a binomial tree must be replaced

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by a ”multidimensional” binomial tree, also known as a *binomial pyramid*, which is feasible for two assets but becomes rather unmanageable for three or more assets. The Longstaff and Schwartz method is able, in principle, to cope with several underlying assets, but in this case it involves complicated Hermite polynomials. Moreover, as with all Monte Carlo methods, it becomes quite slow when the number of assets increases. So there is an obvious need for a fast, accurate and simple method to value and hedge American basket and spread options. In this paper we provide such a method.

In Borovkova et al. (2007) a new approach was introduced for valuing and hedging of European basket and spread options: the so-called *GLN (Generalized LogNormal) method*, which subsequently has been extended to Asian basket options. In this paper we adopt the main ideas of the GLN method and extend it for building a single binomial tree for the basket value evolution. With such a tree, American (and other, e.g., Bermudan) basket options can be priced and hedged. First, we briefly summarize the GLN method.

2. The GLN approach

When valuing basket options, we have to deal with the distribution of the basket value on some date, such as the option’s maturity date. In Black-Scholes model, this distribution is lognormal. In case of a basket of assets, this distribution is no longer lognormal, even when the distributions of all the assets are. It is observed that the sum of lognormal random variables can be well approximated by again a lognormal random variable, who’s moments can be matched to the moments of the sum. However, for spreads and, more general, baskets with negative and positive weights (i.e., portfolios containing long *and* short position) this is no longer possible, because such a basket can have negative values and the distribution of such a basket can be negatively skewed. A three parameter family of the so-called *generalized lognormal distributions* offers a convenient way out: a distribution from this family can have negative values (the so-called *shifted lognormal*), negative skewness (the *negative lognormal*) or both (the *negative shifted lognormal*). For illustration of these distributions, see Figure 1. To apply the moment-matching procedure, we have to match not

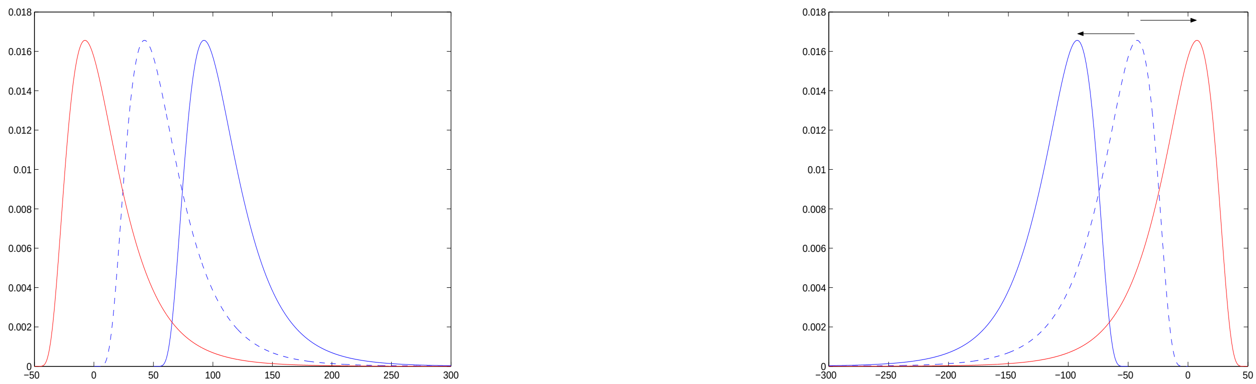

Figure 1: Shifted and negative shifted lognormal densities.

only the first two but the first three moments of the basket, as the generalized lognormal family contains three free parameters which we will call m, s and τ: respectively the scale, shape and shift parameters.

Assume that the N assets in the basket are futures (e.g., commodity futures) following, under the risk-neutral measure, drift-free correlated Geometric Brownian motions, with initial values $F_i(0)$, volatilities σ_i and correlations ρ_{ij}, and let the portfolio weights be a_i (which can be negative or positive). Then, on date T, the first three moments of the basket distribution can be easily calculated:

$$M_1(T) \quad = EB(T) \quad = \sum_{i=1}^{N} a_i F_i(0), \tag{1}$$

$$M_2(T) \quad = EB^2(T) \quad = \sum_{j=1}^{N}\sum_{i=1}^{N} a_i a_j F_i(0) F_j(0) e^{(\rho_{i,j}\sigma_i\sigma_j T)}, \tag{2}$$

$$M_3(T) \quad = EB^3(T) \quad = \sum_{k=1}^{N}\sum_{j=1}^{N}\sum_{i=1}^{N} a_i a_j a_k F_i(0) F_j(0) F_k(0) e^{[(\rho_{i,j}\sigma_i\sigma_j + \rho_{i,k}\sigma_i\sigma_k + \rho_{j,k}\sigma_j\sigma_k)T]}, \tag{3}$$

and the basket skewness is

$$\eta_{B(T)} = \frac{E\big[B(T) - EB(T)\big]^3}{s^3_{B(T)}},$$

where $s_{B(T)} = \sqrt{EB^2(T) - (EB(T))^2}$ is the standard deviation of the basket value at time T. The first three moments for the shifted lognormal distribution are

$$M_1 \quad = \quad \tau + \exp\big(m + \frac{1}{2}s^2\big), \tag{4}$$

$$M_2 \quad = \quad \tau^2 + 2\tau \exp\big(m + \frac{1}{2}s^2\big) + \exp\big(2m + 2s^2\big), \tag{5}$$

$$M_3 \quad = \quad \tau^3 + 3\tau^2 \exp\big(m + \frac{1}{2}s^2\big) + 3\tau \exp\big(2m + 2s^2\big) + \exp\big(3m + \frac{9}{2}s^2\big), \tag{6}$$

and for negative shifted lognormal distribution, the moments M_1 and M_3 are replaced by $-M_1$ and $-M_3$. So, depending on the sign of $\eta_{B(T)}$, we set the first three moments of the appropriate lognormal distribution (shifted if $\eta_{B(T)} > 0$ and negative shifted if $\eta_{B(T)} < 0$) equal to the first three moments of the basket and numerically solve this system of equations for m, s and τ.

The next step involves replacing our basket $B(T)$ by $\tilde{B}(T)$, which equals either $B(T) - \tau$ or $-B(T) - \tau$, depending on the sign of $\eta_{B(T)}$, and observing that $\tilde{B}(T)$ has the regular lognormal distribution. Now, the (European) basket option on $B(T)$ can be valued as the option on $\tilde{B}(T)$ by Black-Scholes formula, by adjusting the strike to $K - \tau$ and, in case of $\eta_{B(T)} < 0$, replacing a call by a put and vice versa.

When applying the GLN method to American options, we are facing an extra complication: we need to approximate not just the distribution of the terminal value of the basket $B(T)$ (or of the average, as in the case of an Asian option), but the entire basket value *process* $(B(t))_{t=0}^{T}$, from the option's initiation ($t = 0$) until the option's maturity T. For that, we replace the basket process $B(t)$ by $\tilde{B}(t)$, which equals either $B(t) - \tau(t)$ or $-B(t) - \tau(t)$, depending on the sign of the basket's skewness, and then approximating the process $\tilde{B}(t)$ by an appropriate Geometric Brownian motion. The next section explains these ideas in more detail.

3. The GLN approach for American basket options

The main difficulty in pricing an American basket option by a binomial method is the high dimensionality of the binomial tree. Here we adopt the main ideas of the GLN approach for building a single binomial tree for the basket's evolution, such as those routinely used in valuing single-asset American options. Such a binomial tree can be used for pricing and hedging American (and other, such as Bermudan) basket options.

Consider a basket option maturing at time T, and let $0 = t_0 < t_1 < t_2 < \cdots < t_{i-1} < t_i < \ldots < t_n = T$ be equidistant discrete partition of the interval $[0, T]$. At any time t_i, we can compute the moments and the skewness of the basket and employ the moment matching procedure outlined above. However, the parameters of the approximating distribution, namely $m(t)$, $s(t)$ and $\tau(t)$, are varying in time. So first we empirically investigate the variability of these parameters.

We show this on the example of the following basket (spread): F_0=[100;120], σ=[0.2;0.3], $\rho_{1,2}$=0.9, a=[-1;1] and T=1 year. We divide interval [0,T] into 25 equal time steps, calculate the first three moments and skewness of basket, choose the suitable approximation distribution and match its parameters for every discrete time $(t_i)_{i=0}^{n}$. We find in this case (and for all other numerous baskets we considered) that sign of the basket skewness does not change in time. In this case it remains positive. So we can choose the same kind of approximating distribution over whole interval [0,T]. The plots of the parameters $m(t)$, $s(t)$ and $\tau(t)$ vs. time are shown in Figure 2.

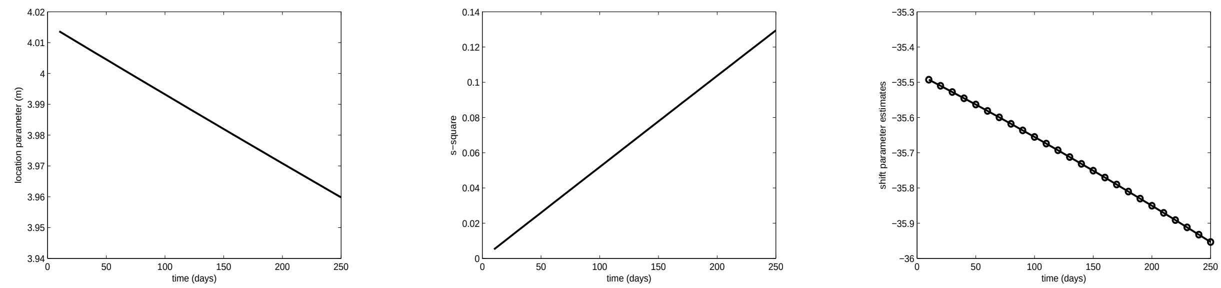

Figure 2: Parameters $m(t_i), s^2(t_i), \tau(t_i)$ vs. time t_i

Note that, for this example, the maximum discrepancy between $(\tau(t_i))_{i=0}^{25}$ over interval [0,1] is relatively small, less than 1% of its average value over the entire interval. This has been also observed for all other basket examples we considered and maturities up to one year (which is generally longer than most of realistic traded basket options). So for simplification, we can take the shift parameter τ constant and equal to the average value over all t_i's.

Define

$$B^*(t) = B(t) - \tau(t) \quad \text{or} \quad B^*(t) = -B(t) - \tau(t), \tag{7}$$

depending on the approximating distribution (shifted resp. negative shifted lognormal). (As noted above, we can take constant τ, or shift the basket value at each discrete time step t_i by the parameter $\tau(t_i)$ obtained for that time). Observe that, for any t, $B^*(t)$ has the lognormal distribution with parameters $(m(t),\ s(t))$. Assume that $B^*(t)$ satisfies the following stochastic differential equation:

$$dB^*(t) = \mu^* B^*(t)dt + \sigma^* B^*(t)dW(t), \tag{8}$$

where $W(t)$ is the standard Brownian motion. Note that $\text{Var}(\log B^*(t)) = \sigma^{*2}t = s^2(t)$, so the volatility σ^* can be computed as

$$\sigma^* = \sqrt{\frac{s^2(t)}{t}}. \tag{9}$$

For σ^* to be constant, $s^2(t)$ must be linear function of time. This is exactly what we observe empirically, in Figure 2 and in all other considered examples. Furthermore, note that

$$E(\log B^*(t)) = m(t) = \log B^*(0) + (\mu^* - \frac{1}{2}\sigma^{*2})t, \tag{10}$$

from which there follows the expression for μ^*:

$$\mu^* = \frac{m(t) - \log B^*(0)}{t} + \frac{1}{2}\sigma^{*2}. \tag{11}$$

For μ^* to be constant, the parameter $m(t)$ must be a linear function of time with the intercept equal to $\log B^*(0)$, which is indeed exactly the case, as shown in Figure 2.

Now we can build the binomial tree for our basket $B(t)$ by the following algorithm.

- Build the binomial tree for the value $B^*(t)$, which follows GBM. At each time step on the tree, the value $B*$ moves up to uB^* with probability q or down to dB^* with probability $1-q$, where

$$\begin{aligned} u &= \exp\left(\left(\mu^* - 0.5\sigma^{*2}\right)\Delta t + \sigma\sqrt{\Delta t})\right), \\ d &= \exp\left(\left(\mu^* - 0.5\sigma^{*2}\right)\Delta t - \sigma\sqrt{\Delta t})\right), \\ q &= \frac{\exp\left(\mu^*\Delta t\right) - d}{u - d}. \end{aligned}$$

- Translate the obtianed binomial tree for B^* into the binomial tree for B using equations

$$\begin{aligned} B(t) &= B^*(t) - \tau(t), && \text{if the approximating distribution is shifted lognormal,} \\ B(t) &= -B^*(t) - \tau(t), && \text{if the approximating distribution is negative shifted lognormal.} \end{aligned}$$

- This tree can be now used for valuing an American option on $B(t)$, computing option's deltas at each tree node and deciding on early exercise.

4. Numerical study

We apply the binomial tree model to several basket options. We compare the obtained prices to those obtained by other existing models, wherever possible.

Consider call and put options on the following baskets:

Basket 1: F_0=[50;50]; σ=[0.3;0.2]; $\rho_{1,2}$=0.6; a= [0.3;0.7]; X=50.

Basket 2: F_0=[100;120]; σ=[0.2;0.3]; $\rho_{1,2}$=0.9; a=[-1;1]; X=30.

Basket 3: F_0=[150;100]; σ=[0.3;0.2]; $\rho_{1,2}$=0.7; a=[-1;1]; X=-40.

Basket 4: F_0=[95;90;105]; σ=[0.2;0.3;0.25]; $\rho_{1,2}$=$\rho_{1,2}$=0.9; $\rho_{2,3}$=0.8; a=[1;-0.8;-0.5]; X=-30.

Basket 5: F_0=[100;90;95]; σ=[0.25;0.3;0.2]; $\rho_{1,2}$=$\rho_{1,2}$=0.9; $\rho_{2,3}$=0.8; a=[0.6;0.8;-1]; X=40.

We assume the risk-free interest rate is 5 % per annum and the time to expiry (T) is 1 year.

The American call and put basket option prices obtained by the GLN method-based binomial tree are given in columns 5 and 8 in Table 1. Baskets 1, 2 and 3 consist of two assets. For these baskets, we compare the prices obtained by our method (we call it *GLN-based binomial tree*) to those obtained by the two-dimensional binomial tree, or binomial pyramid (BP) method. Basket 1 is a basket with positive weights. In that case, the implied binomial tree approach of Rubinstein (1994) is applicable, so we compare the prices obtained by our method to the implied tree prices as well. Baskets 4 and 5 contain three assets. Multidimensional binomial tree method can deal with option on three assets, but this leads to a very large and unpractical binomial tree, so the results of this method have not been included.

Table 1: American basket option prices

Basket	approx. distr.	American call option			American put option		
		BP	IBT	**GLN BT**	BP	IBT	**GLN BT**
Basket 1	shifted	3.9672	3.9698	**3.9749**	3.9676	3.9584	**3.9751**
Basket 2	shifted	4.3799	—	**4.3733**	14.0781	—	**14.0748**
Basket 3	neg. shifted	8.2335	—	**8.2593**	17.9211	—	**17.9469**
Basket 4	neg. shifted	—	—	**7.6698**	—	—	**7.1857**
Basket 5	shifted	—	—	**6.8761**	—	—	**9.7825**

Table 1 shows that the GLN binomial tree approach performs remarkably well in term of option pricing. The prices obtained by our method are close to those obtained by the full binomial tree method, for basket with positive and negative weights and for baskets with positive and negative skewness. In case of basket with positive weights, its performance is also comparable to the implied binomial tree method.

The main attraction of our proposed method is that it reduces high dimensionality of the full binomial tree, as the implied binomial tree method does, but it is also applicable to baskets with negative weights, such as spreads - something the method of implied binomial tree cannot cope with.

In the full version of the paper we shall also demonstrate that the delta-hedging performance of our method is remarkably accurate, on the basis of both simulated and real market asset prices.

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