

Asian basket options and implied correlations in energy markets

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Abstract

We address the problem of valuation and hedging of Asian basket and spread options - derivatives common in energy markets. We extend the Generalized LogNormal approach, introduced in Borovkova et al. (2007), to Asian basket options and apply it to energy option markets. We provide closed form formulae for the option price and the greeks, which is extremely useful for option traders. Inverting the option pricing formula allows us to imply the correlation between the assets in the spread from liquid spread option prices. Numerical simulations and the application to energy markets show that our approach performs remarkably well in terms of option pricing and delta hedging. We analyze the option's sensitivity to volatilities and correlations, and demonstrate that the implied correlation between NYMEX crude oil and heating oil prices shows the behavior similar to the implied volatility skew.

Keywords: basket and spread options, Asian options, option greeks, oil futures, implied correlation.

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1 Introduction

Energy companies typically hold portfolios of several energy commodities. Often, these portfolios simultaneously contain long and short commodity positions. For example, a refinery purchases crude oil and sells refined products such as heating oil and unleaded gasoline. The refinery's exposure is to the so-called *crack spread*, which is the difference between the prices of the raw material (crude oil) and refined products in appropriate proportions³. Another example is a power generating plant, which buys fuel (gas or coal) (and possibly emission rights) and sells electricity. For such plant the exposure is to the so-called *spark* or *dark spread*, which is the difference between the fuel price (and eventual emission rights) and the price of generated electricity.

To hedge their price exposure, such companies may enter futures transactions on the both long and short sides of their portfolios. However, hedging with options would allow to profit from a possible upside, while limiting the downside risk. An optimal hedge would be achieved in this case by a *basket* or *spread option*: an option whose underlying asset is a basket or a spread, i.e., a portfolio of several assets, possibly containing long and short positions. An appropriate basket or spread option allows managing the market risk of the entire commodity portfolio using just one derivative. Basket options are fundamentally different from a collection of options on individual assets comprising the basket, as they are the so-called *correlation*, or *cross-commodity derivatives* that allow to manage the *correlation risk*.

Spread options are very common in energy markets, they are traded both over-the-counter and on commodity exchanges such as ICE⁴ and NYMEX. However, most traded options in energy markets are Asian options (so not European-style), i.e., their payoff is based on the *average* (and not the terminal) price of the underlying asset. The reason for this is two-fold: first, volatilities of energy prices are much higher than volatilities of stocks or financial indices. Asian options are cheaper than their European counterparts, since the volatility is reduced by averaging the underlying prices. Second, most delivery contracts in energy markets are based on average commodity prices over a specified period. These considerations gave rise to popular over-the-counter derivatives: Asian basket and spread options. Nowadays, such options are also actively traded on exchanges⁵.

The fundamental difficulty in pricing both Asian and basket options is determining the distribution of the sum or the average of underlying asset prices. Even in the Black-Scholes framework (where the prices are assumed to follow the Geometric Brownian motion, so have the lognormal distribution), this difficulty is not easily resolved, as the sum (or the arithmetic average) of lognormal random variables is not lognormally distributed. Several approximation methods have been proposed for both Asian and basket options: Kemna and Vorst (1990) proposed approximating the arithmetic average by the geometric average; Turnbull and Wakeman (1991) proposed approximating the basket value or the arithmetic price average by the lognormal distribution, matching the first two moments. However, these methods can deal only with averages or baskets with positive weights. The value of a basket containing also short positions (such as a spread) can be negative, and its distribution can be negatively skewed, so the lognormal distribution cannot be applied even as an approximation.

Several methods have been proposed for valuing European spread options, for example, Kirk

³It is also often called the *3:2:1 crack spread*, as 3 parts of crude oil produce approximately 2 parts of heating oil and 1 part of gasoline.

⁴InterContinental Exchange

⁵For example, the NYMEX 1:1 crude oil/heating oil crack spread options are Asian-style.

(1995) (who was inspired by the classical paper of Margrabe (1978)) replaced the difference of asset prices by their ratio and adjusted the strike price. This approach, although giving a good approximation for the spread option price, can only deal with spreads between *two* asset prices, while many spreads in energy markets (e.g., crack spread or clean spark and dark spreads) consist of more than two asset prices. The Asian exercise feature of energy spread and basket options presents an extra difficulty in their valuation and hedging. Currently there is no method other than Monte Carlo simulations for pricing Asian basket and spread options. The Monte Carlo method can be very slow in this case, as it involves generating simultaneous price paths for several correlated assets. With this paper, we fill this substantial gap in option pricing literature and provide an analytic solution to this option pricing problem.

In Borovkova et al. (2007), the Generalized Lognormal (GLN) approach was proposed for valuing European basket and spread options. It is a moment-matching method, based on approximating the spread or basket distribution by a Generalized Lognormal family of distributions, which allows for negative values and negative skewness. The GLN approach is able to produce closed-form formulae for the option price and the greeks, which can then be quickly and accurately evaluated: something that practitioners value greatly.

In this paper we use the ideas of the GLN approach to price and hedge Asian basket options. We obtain the closed form expression for the option price, which allows us to imply correlations between commodities from exchange-listed prices of spread or basket options, even when these are Asian-style. We apply the GLN approach to the NYMEX heating oil/crude oil crack spread Asian options, to estimate and analyze the implied correlation.

The paper is organized as follows. In Section 2 we briefly outline the GLN approach and obtain prices and greeks for Asian basket options. In Section 3 we assess the performance of our method on the basis of simulations. In Section 4 we apply the method to the oil market data and analyze the implied correlation between NYMEX crude oil and NYMEX heating oil futures from the NYMEX crude oil/heating oil crack spread option prices. Conclusions and suggestions for future research are given in final section.

2 Asian basket options

2.1 The GLN approach

Let N be the number of assets in a basket and $(a_i)_{i=1}^N$ be the collection of (possibly negative) portfolio weights (for example, the weights for 3:2:1 crack spread are $(3, -2, -1)$). As in most traded basket and spread options, the assets in the basket are assumed to be *commodity futures*, whose prices at time t we denote $(F_i(t))_{i=1}^N$. The basket value at the time t is thus

$$B(t) = \sum_{i=1}^N a_i F_i(t).$$

Furthermore, we assume that under the pricing measure, the futures prices follow correlated zero-drift Geometric Brownian Motions:

$$\frac{dF_i(t)}{F_i(t)} = \sigma_i dW_i(t), \quad i = 1, 2, \dots, N, \quad (1)$$

where σ_i is the volatility of futures i (assumed constant), $W_i(t)$ and $W_j(t)$ are the Brownian motions driving the futures i and j , with correlation $\rho_{i,j}$ (i.e., $dW_i(t)dW_j(t) = \rho_{i,j}dt$). The payoff of a European basket call option is given by

$$(B(T) - X)^+,$$

where $B(T)$ is the terminal basket value at the time of maturity T , and X is the strike price. If assets in the basket are futures, whose prices $(F_i(t))_{i=1}^N$ follow (1), then it can be easily shown that the first three moments of $B(T)$ are given by

$$M_1(T) = EB(T) = \sum_{i=1}^N a_i F_i(0), \quad (2)$$

$$M_2(T) = EB^2(T) = \sum_{j=1}^N \sum_{i=1}^N a_i a_j F_i(0) F_j(0) e^{(\rho_{i,j} \sigma_i \sigma_j T)}, \quad (3)$$

$$M_3(T) = EB^3(T) = \sum_{k=1}^N \sum_{j=1}^N \sum_{i=1}^N a_i a_j a_k F_i(0) F_j(0) F_k(0) e^{[(\rho_{i,j} \sigma_i \sigma_j + \rho_{i,k} \sigma_i \sigma_k + \rho_{j,k} \sigma_j \sigma_k) T]}. \quad (4)$$

In terms of the first three moments, the skewness of basket can be calculated as

$$\eta_{B(T)} = \frac{E[B(T) - EB(T)]^3}{s_{B(T)}^3},$$

where $s_{B(T)} = \sqrt{EB^2(T) - (EB(T))^2}$ is the standard deviation of the basket value at time T .

The stochastic differential equation (1) implies that the distribution of the futures prices is lognormal. Although the sum of lognormal random variables is not lognormal, studies from various areas of science have shown that lognormal approximation of the distribution of such sums works very well (Mitchell (1968), Aitchison and Brown (1957), Crow and Shimizu (1988), Limpert et al. (2001)), and certainly better than the normal approximation. Recall that here we consider baskets with possibly negative weights, such as spreads. Hence, we cannot approximate the distribution of $B(T)$ by a lognormal distribution, since such a basket can have negative values or negative skewness. However, a more general three-parameter family of lognormal distributions: shifted and negative shifted lognormal, can be used to approximate the distribution of a general basket. The shifted lognormal distribution is obtained by shifting the regular lognormal density by a fixed amount along the x -axis, and the negative lognormal - by reflecting the lognormal density across the y -axis. The negative shifted lognormal distribution is the combination of the negative and the shifted one. Figure 1 shows the densities of these two distributions. Note that this family of distributions is flexible enough to incorporate negative values and negative skewness: something that the regular lognormal distribution is unable to do.

For each of these distributions, we can easily derive the first three moments. For example, for the shifted lognormal distribution, these moments are given by

$$M_1 = \tau + \exp\left(m + \frac{1}{2}s^2\right), \quad (5)$$

$$M_2 = \tau^2 + 2\tau \exp\left(m + \frac{1}{2}s^2\right) + \exp\left(2m + 2s^2\right), \quad (6)$$

$$M_3 = \tau^3 + 3\tau^2 \exp\left(m + \frac{1}{2}s^2\right) + 3\tau \exp\left(2m + 2s^2\right) + \exp\left(3m + \frac{9}{2}s^2\right). \quad (7)$$

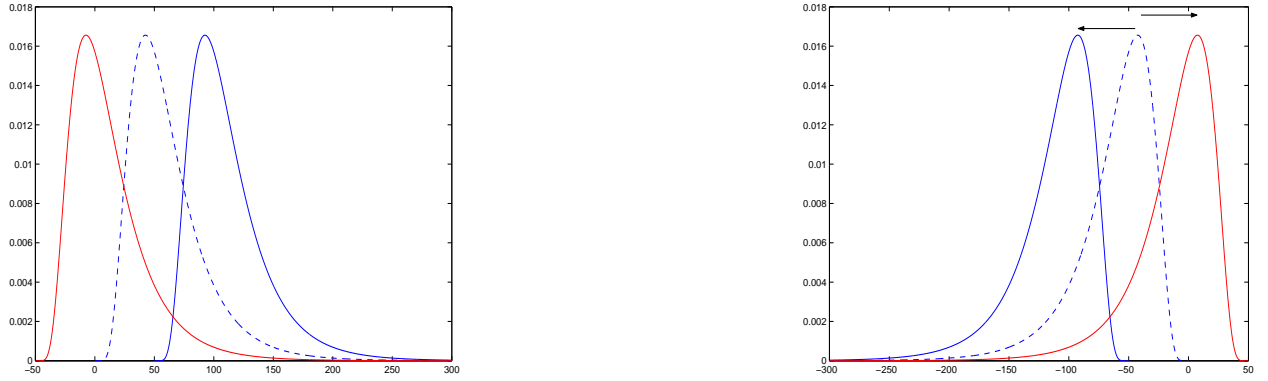


Figure 1: Shifted and negative shifted lognormal densities.

where m, s and τ are respectively the scale, shape and shift parameters. The GLN approach is a moment-matching method: the parameters of the approximating distribution, for example the shifted lognormal, are estimated by matching the first three moments of the basket ((2), (3), and (4)) with the first three moments of the shifted lognormal distribution ((5), (6), and (7)), by solving a nonlinear equation system with three unknown parameters m, s and τ . Such a solution exists and is unique; the proof of this we omit in this paper. Note that, for the negative shifted lognormal approximation, the moments M_1 and M_3 are replaced in the moment-matching procedure by respectively $-M_1$ and $-M_3$. The question which lognormal distribution should be used in each particular case (shifted or negative shifted) is resolved by computing and examining the basket distribution's skewness. If it is positive, the shifted lognormal approximation is used, otherwise the negative shifted lognormal.

Having chosen the appropriate distribution from the lognormal family, we can derive closed formula for a basket option price by applying the Black-Scholes formula as follows. Let the terminal value (B_A) of the basket A be lognormally distributed with parameters m, s . Furthermore, let the basket B with value B_B have the following relationship with the basket A:

$$B_B(T) = B_A(T) + \tau,$$

where τ is a constant. Then the distribution of B_B must be shifted lognormal with the parameters m, s, τ . On the maturity date T , the payoff of a call option on the basket B is

$$(B_B(T) - X)^+ = ((B_A(T) + \tau) - X)^+ = (B_A(T) - (X - \tau))^+.$$

This is the payoff of a call option on the basket A with the same maturity date T and the strike price $(X - \tau)$, and such a call option can be valued by the Black-Scholes formula.

Next, suppose that the basket C with value B_C has the following relationship to the basket A:

$$B_C(T) = -B_A(T).$$

The distribution of the terminal basket value B_C must be negative lognormal with parameters m, s . On the maturity date T , the payoff of a call option on the basket C is

$$(B_C(T) - X)^+ = ((-X) - B_A(T))^+.$$

This is the payoff of a *put option* on the basket A, with the maturity date T and the strike price $(-X)$, and to value such a put option, the Black-Scholes formula can be applied again. Valuation of a basket option using the negative shifted lognormal distribution can be seen as a combination of the shifted and negative lognormal distributions. The closed form expressions for the European basket call option price for the two approximating distributions are given in Borovkova et al. (2007).

2.2 Asian basket option: the price

Generalized Lognormal Method naturally extends to the case of Asian basket options, as we shall now demonstrate. We again consider a call option on a basket of futures, only now with an Asian payoff:

$$(A_B(T) - X)^+,$$

where $A_B(T)$ is the daily arithmetic average value of the basket over the period between today ($t_0 = 0$) and the time of maturity T , and X is the strike price. Valuation of an Asian option should be considered for two cases:

1. The averaging period starts today or at some later date (*newly issued option*).
2. The averaging period has already started (*already issued option*).

First, consider newly issued options (at a later stage we show how our approach extends to already issued options). We assume that the averaging starts today ($t_0 = 0$) or on some later date $t_{n_1} \geq t_0$ and ends on the maturity date $T = t_{n_2} > t_{n_1}$. Denote the arithmetic average of the i -th futures price by

$$A_i(T) = \frac{1}{n} \sum_{k=n_1}^{n_2} F_i(t_k), \quad i = 1, 2, \dots, N, \quad (8)$$

where $n = n_2 - n_1 + 1$. It can be easily shown that the first two moments of $A_i(T)$ are given by

$$M_{1,i}(T) = F_i(0), \quad (9)$$

$$M_{2,i}(T) = \frac{1}{n^2} \sum_{p_2=n_1}^{n_2} \sum_{p_1=n_1}^{n_2} (F_i(0))^2 \exp(\sigma_i^2 \min(t_{p_1}, t_{p_2})). \quad (10)$$

Since arithmetic averages of the individual asset prices $A_i(T)$, $i = 1, \dots, N$, are always positive (and the prices are assumed to be lognormal), we can approximate the distribution of $A_i(T)$ by the regular lognormal distribution, as in Turnbull and Wakeman (1991). By matching the first two moments, the parameters of the lognormal approximating distribution can be taken

$$\lambda_i = \log \left(\frac{M_{1,i}^2}{\sqrt{M_{2,i}}} \right), \quad (11)$$

$$\gamma_i = \sqrt{\log \left(\frac{M_{2,i}}{M_{1,i}^2} \right)}, \quad (12)$$

where λ_i is the scale and γ_i the shape parameters.

The average basket value at T is

$$A_B(T) = \frac{1}{n} \sum_{k=n_1}^{n_2} B(t_k) = \frac{1}{n} \sum_{k=n_1}^{n_2} \sum_{i=1}^N a_i F_i(t_k) = \sum_{i=1}^N a_i A_i(T).$$

Hence, the average value of the basket at date T is simply the basket of individual assets' averages, with the same weights. Note that, even though all individual asset's averages are positive, the basket's average $A_B(T)$ can be negative because of possibly negative weights a_i . Its distribution can also be negatively skewed. Hence, the lognormal approximation cannot be applied to $A_B(T)$, while it can be applied to $A_i(T)$'s. So we again use the generalized lognormal family to approximate the distribution of $A_B(T)$, as in the original GLN method.

If the correlation between the i th and j th log-futures prices is $\rho_{i,j}$, then the correlation between the logarithms of their geometric averages is also $\rho_{i,j}$. This is not true for the logarithms of arithmetic averages. However, the arithmetic average can be well approximated by the geometric average, hence the correlation between the logarithms of the arithmetic averages of these futures prices should also be well approximated by $\rho_{i,j}$. Under this approximation, the first three moments of the average basket value can be calculated as follows:

$$\bar{M}_1(T) = EA_B(T) = \sum_{i=1}^N a_i F_i(0), \quad (13)$$

$$\bar{M}_2(T) = EA_B^2(T) = \sum_{j=1}^N \sum_{i=1}^N a_i a_j F_i(0) F_j(0) e^{(\rho_{i,j} \gamma_i \gamma_j)}, \quad (14)$$

$$\bar{M}_3(T) = EA_B^3(T) = \sum_{k=1}^N \sum_{j=1}^N \sum_{i=1}^N a_i a_j a_k F_i(0) F_j(0) F_k(0) e^{(\rho_{i,j} \gamma_i \gamma_j + \rho_{i,k} \gamma_i \gamma_k + \rho_{j,k} \gamma_j \gamma_k)}. \quad (15)$$

As before, the skewness of the average basket value can be calculated as

$$\eta_{A_B(T)} = \frac{E[A_B(T) - EA_B(T)]^3}{s_{A_B(T)}^3}, \quad (16)$$

where $s_{A_B(T)} = \sqrt{EA_B^2(T) - (EA_B(T))^2}$ is the standard deviation of the average basket value at time T .

We now approximate the distribution of the average basket value using a family of lognormal distributions: shifted and negative shifted, by matching the first three moments of the average basket value \bar{M}_1 , \bar{M}_2 and \bar{M}_3 (given in (13), (14) and (15)) with the first three moments of the appropriate approximating distribution (e.g. those given in (5)-(7) for the shifted lognormal distribution), and solve this nonlinear system of three equation for the three unknown parameters m , s and τ . The approximating distribution is chosen on the basis of the skewness, as before.

The valuation of an Asian basket option is now similar to a European basket option: for the shifted lognormal approximation, we replace the strike price X by $(X - \tau)$ and for the negative shifted lognormal, we also multiply the strike with -1 and replace the call with the put and vice versa. In both cases, we have reduced the problem to option pricing on lognormal underlying value, hence, the Black-Scholes formula can be applied. This leads to the closed form expressions

for the (approximate) price of an Asian basket call option c_A , given by the following formulae. Everywhere $\bar{M}_1(T)$ and $\bar{M}_2(T)$ denote the first two moments of the average basket value on the maturity date T , given by (13) and (14), $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution, and $d_2 = d_1 - V$.

- Using the shifted lognormal approximation

$$c_A = e^{-rT} [(\bar{M}_1(T) - \tau)\Phi(d_1) - (X - \tau)\Phi(d_2)] \quad (17)$$

$$\text{where } d_1 = \frac{\log(\bar{M}_1(T) - \tau) - \log(X - \tau) + \frac{1}{2}V^2}{V}$$

$$V = \sqrt{\log\left(\frac{\bar{M}_2(T) - 2\tau\bar{M}_1(T) + \tau^2}{(\bar{M}_1(T) - \tau)^2}\right)}$$

- Using the negative shifted lognormal approximation

$$c_A = e^{-rT} [(-X - \tau)\Phi(-d_2) + (\bar{M}_1(T) + \tau)\Phi(-d_1)] \quad (18)$$

$$\text{where } d_1 = \frac{\log(-\bar{M}_1(T) - \tau) - \log(-X - \tau) + \frac{1}{2}V^2}{V}$$

$$V = \sqrt{\log\left(\frac{\bar{M}_2(T) + 2\tau\bar{M}_1(T) + \tau^2}{(\bar{M}_1(T) + \tau)^2}\right)}$$

The above considerations can be easily extended to an already issued option. Suppose that the arithmetic average of the i -th futures price is, as before,

$$A_i(T) = \frac{1}{n} \sum_{k=n_1}^{n_2} F_i(t_k),$$

where the prices $F_i(t_{n_1}), F_i(t_{n_1+1}), F_i(t_{n_1+2}), \dots, F_i(t_m)$ have been already observed. We decompose $A_i(T)$ into:

$$\begin{aligned} A_i(T) &= \frac{n_1^*}{n} A_{1,i}(T) + \frac{n_2^*}{n} A_{2,i}(T) \\ &= \frac{n_1^*}{n} \left[\frac{1}{n_1^*} \sum_{k=n_1}^m F_i(t_k) \right] + \frac{n_2^*}{n} \left[\frac{1}{n_2^*} \sum_{k=m+1}^{n_2} F_i(t_k) \right], \end{aligned}$$

where $n_1^* = m - n_1 + 1$, $n_2^* = n_2 - m$. Here $A_{1,i}(T)$ and $A_{2,i}(T)$ denote the arithmetic average of the i -th futures price over prices that have been already observed and over the future period, respectively. In terms of $A_{1,i}(T)$ and $A_{2,i}(T)$, the average basket value $A_B(T)$ can be decomposed into:

$$\begin{aligned} A_B(T) &= \frac{n_1^*}{n} A_{B,1}(T) + \frac{n_2^*}{n} A_{B,2}(T) \\ &= \sum_{i=1}^N a_i A_{1,i}(T) + \sum_{i=1}^N a_i A_{2,i}(T) \end{aligned}$$

As a result, the payoff of call Asian basket option is given by:

$$\begin{aligned} (A_B(T) - X)^+ &= \left(\frac{n_1^*}{n} A_{B,1}(T) + \frac{n_2^*}{n} A_{B,2}(T) - X \right)^+ \\ &= \frac{n_2^*}{n} (A_{B,2}(T) - X^*)^+, \end{aligned}$$

where $X^* = \frac{n}{n_2^*} X - \frac{n_1^*}{n_2^*} A_{B,1}(T)$. Hence, an already issued option can be valued as a newly issued option, by changing the strike price X to X^* and multiplying the result by $\frac{n_2^*}{n}$.

2.3 Asian basket option: the greeks

Delta hedging of an option plays the crucial role in managing risks associated with an option portfolio. Hence, providing the closed form expressions for the deltas and other option's greeks, which can be quickly and accurately evaluated, is essential for option traders and other market participants. One of the main attractive features of our approach is its ability to provide (approximate) closed formulae for option's deltas and other greeks, and not just the option price, as in Monte Carlo method.

Recall that the option's greeks are the partial derivatives of the option price with respect to the parameters such as the prices of the underlyings, volatilities, correlations, time to maturity and the interest rate. Here we are particularly interested in basket option's deltas and vegas, i.e. the sensitivities of the basket option price with respect to the futures prices, their volatilities and correlations. The basket option prices, given in (17)-(18), contain these parameters explicitly within the moments \bar{M}_1 , \bar{M}_2 and \bar{M}_3 , but also implicitly within the shift parameter τ . Hence, the main difficulty in obtaining the correct formulae for option's greeks is finding explicit expressions for the first derivatives of τ with respect to F_i , σ_i and $\rho_{i,j}$. For example, to obtain the option's deltas, we must differentiate the option price with respect to F_i , $i = 1, \dots, N$. The resulting derivative contains the partial derivative $\partial\tau/\partial F_i$. Although there is no closed formula for τ in terms of the model parameters, it turns out we can obtain such formulae for the *partial derivatives* of τ with respect to F_i , σ_i and $\rho_{i,j}$.

Suppose that we approximate the basket distribution by the shifted lognormal distribution by the three-moments matching procedure. We can differentiate the equations in the nonlinear equation system with respect to the futures price F_i . This differentiation forms a *linear* equation system

$$\begin{bmatrix} 1 & \alpha & s\alpha \\ 2(\tau + \alpha) & 2(\tau\alpha + \beta) & 2s(\tau\alpha + 2\beta) \\ 3(\tau^2 + 2\tau\alpha + \beta) & 3(\tau^2\alpha + 2\tau\beta + \theta) & 3s(\tau^2\alpha + 4\tau\beta + 3\theta) \end{bmatrix} \times \begin{bmatrix} \partial\tau/\partial F_i \\ \partial m/\partial F_i \\ \partial s/\partial F_i \end{bmatrix} = \begin{bmatrix} \partial\bar{M}_1/\partial F_i \\ \partial\bar{M}_2/\partial F_i \\ \partial\bar{M}_3/\partial F_i \end{bmatrix},$$

where $\alpha = e^{(m+\frac{1}{2}s^2)}$, $\beta = e^{(2m+2s^2)}$, and $\theta = e^{(3m+\frac{9}{2}s^2)}$. Now we can find $\partial\tau/\partial F_i$ by solving the linear equation system above. In the negative shifted lognormal case, the derivative $\partial\bar{M}_1/\partial F_i$ changes to $-\partial\bar{M}_1/\partial F_i$ and $\partial\bar{M}_3/\partial F_i$ to $-\partial\bar{M}_3/\partial F_i$.

To obtain the volatility and correlation vegas, we differentiate the basket option price with respect to all the volatilities σ_i and the correlations $\rho_{i,j}$. The partial derivatives $\partial\tau/\partial\sigma_i$, $\partial\tau/\partial\rho_{i,j}$ are obtained analogously to $\partial\tau/\partial F_i$, i.e. by solving a corresponding linear equation system. The resulting formulae for deltas and vegas of an Asian basket option are reported in Appendix.

In the next two sections we investigate the performance of our generalized lognormal approach in terms of option pricing and delta hedging, by means of simulations and application to the real data from oil markets.

3 Simulation study

We apply the above approach to pricing of Asian options on a number of hypothetical baskets. We choose basket parameters in such a way that both possible approximation distributions occur. The parameters of the test baskets are given in Table 1. Baskets 1, 2 and 4 are spreads, Basket 3 is a "usual" basket, Baskets 5 and 6 are three-asset baskets, with some weights being negative. The interest rate (r) is taken 3% per annum. For all baskets, the options are (almost) at-the-money, the time of maturity (T) is one year (assuming 250 trading days per year) and the averaging period begins on day 101 and ends on day 250. In energy markets, the correlations between commodities in a typical spread are much higher than in equity markets, which is reflected in our choices of ρ 's.

The performance of our approach is investigated by comparing the basket option prices to those obtained by Monte Carlo simulations. For each basket, the Monte Carlo simulation is repeated 1000 times and the price is determined as the mean of these repetitions. The results are given in Table 2. The standard error of the prices obtained by Monte Carlo simulations are given in the parenthesis in the last rows of the tables.

Table 1: Basket parameters

	Basket 1	Basket 2	Basket 3	Basket 4	Basket 5	Basket 6
Futures price ($F(0)$)	[100,120]	[150,100]	[110,90]	[200,60]	[95,90,105]	[100,90,95]
Volatility (σ)	[0.2,0.3]	[0.3,0.2]	[0.3,0.2]	[0.3;0.2]	[0.2,0.3,0.25]	[0.25,0.3,0.2]
Weights (a)	[-1,1]	[-1,1]	[0.7,0.3]	[-1,1]	[1,-0.8,-0.5]	[0.6,0.8,-1]
Correlation (ρ)	$\rho_{1,2}=0.9$	$\rho_{1,2}=0.3$	$\rho_{1,2}=0.9$	$\rho_{1,2}=0.9$	$\rho_{1,2}=\rho_{2,3}=0.9$ $\rho_{1,3}=0.8$	$\rho_{1,2}=\rho_{2,3}=0.9$ $\rho_{1,3}=0.8$
Strike price (X)	20	-50	104	-140	-30	35

Table 2: Prices of Asian basket call options

Method	Basket 1	Basket 2	Basket 3	Basket 4	Basket 5	Basket 6
Skewness (η)	$\eta > 0$	$\eta < 0$	$\eta > 0$	$\eta < 0$	$\eta < 0$	$\eta > 0$
GLN	6.0178 (shifted)	13.1015 (neg. shifted)	8.4178 (shifted)	14.8376 (neg. shifted)	6.0771 (neg. shifted)	7.2401 (shifted)
Monte Carlo	5.9910 (0.0107)	12.9913 (0.0170)	8.3875 (0.0140)	14.8173 (0.0185)	6.0651 (0.0074)	7.2404 (0.0113)

Table 2 shows that our (GLN) approach performs very well: the prices obtained by it are very close to those obtained by Monte Carlo simulations, and are within 95 % Monte Carlo confidence bounds.

We also investigate the performance of the GLN approach on the basis of delta hedging an option and calculating the hedge error. Here we show the results only for Baskets 1 and 5; for other baskets results are similar. The approximating distributions are shifted lognormal for Basket 1, and negative shifted lognormal for Basket 5. We generate 1000 paths of the underlying futures prices, delta-hedge the option on each path and calculate the mean and the standard deviation of the hedge error. This is done for various hedge intervals (1, 5, 10, 15, 20, 30 and 40 days). We plot the ratio of the hedge error standard deviation to the call price vs. the hedge interval in Figures 2 and 3. These plots show that, for both baskets, this ratio decreases together with the hedge interval (the hedge error standard deviation also decreases together with the hedge interval, as we expect). The mean hedge errors are all less than 10% of the option prices.

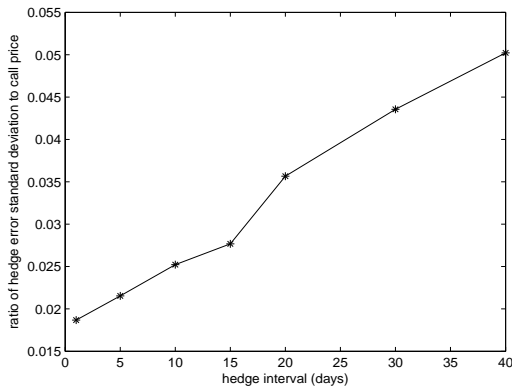


Figure 2: Hedge error vs. hedge interval for Basket 1.

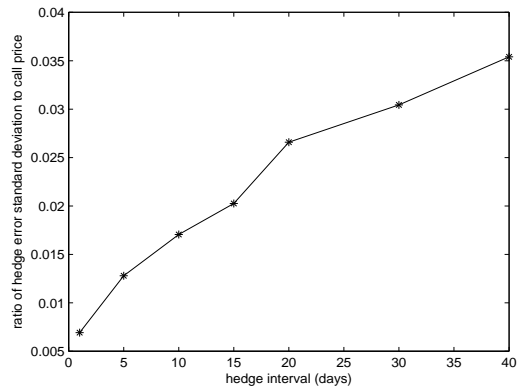


Figure 3: Hedge error vs. hedge interval for Basket 5.

We analyze the correlation vega on the example of an Asian spread option with parameters $F_0 = [100; 120]$, $a = [-1; 1]$, $\sigma = [0.2; 0.3]$. The correlation varies from -1 to 1 and the strike price from 5 to 55. The results are given in Figures 4, 5, and 6. These figures demonstrate the feature of a negative correlation vega for an Asian spread option, that is the price of an Asian spread option decreases as the correlation between the assets increases: the feature shared by a European spread option. The reason for this is that the spread's volatility decreases as the correlation between assets increases, leading to the decrease in the option price.

To analyze the volatility vegas, we apply the GLN approach to an Asian spread option with parameters $F_0 = [100; 120]$, $a = [-1; 1]$, $\rho = 0.9$, $X = 20$. The volatilities σ_1 and σ_2 vary from 5% to 40%. The plots of volatility vegas in Figure 7 show that, for an Asian spread option, volatility vegas can be negative as well as positive. What matters for the spread option price is the spread's volatility, and it can increase or decrease with the individual asset's volatilities.

4 Application to oil markets

We apply the GLN approach to value call options on 1:1 crack spread option maturing in 60 and 90 trading days, both European and Asian style. Note that for Asian style option, the averaging

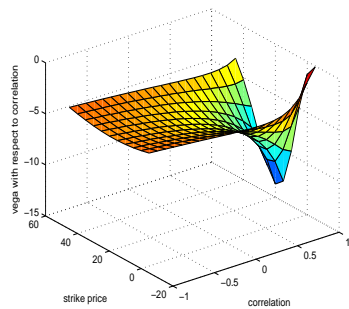


Figure 4: Correlation vega vs. correlations and strike prices

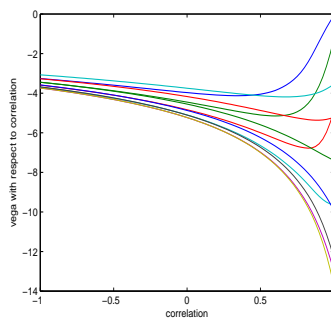


Figure 5: Correlation vega vs. correlations for different strike prices

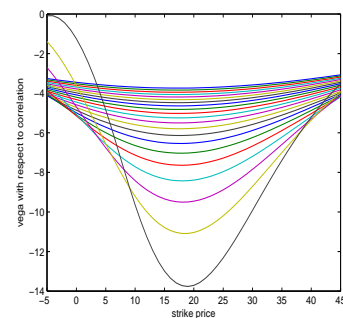


Figure 6: Correlation vega vs. strike prices for different correlations

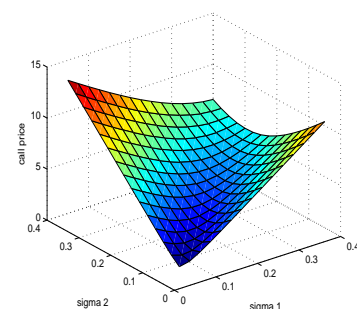
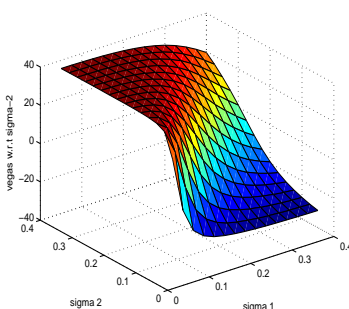
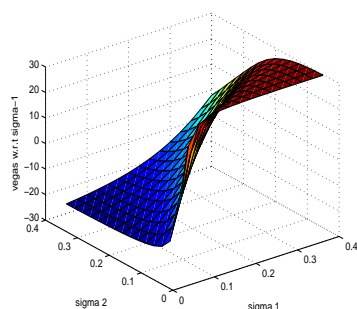


Figure 7: Volatility vegas for different volatilities σ_1 , σ_2 , and call price vs. σ_1 and σ_2

period begins at the first date, and ends at the maturity date. We first evaluate the approach on the basis of delta-hedging, performed on historical data from IPE⁶: Gasoil and Brent crude oil futures prices delivered in December 2002 over the period of June 13, 2001 - November 14, 2002. The hedge is adjusted every day using the appropriate deltas derived by the GLN approach. At the maturity date, we calculate the hedge error, defined as the difference between the discounted cost of hedging the option and the call price obtained by the GLN approach. The relative hedge error (in %) is then the ratio of the hedge error to the call price.

Table 3: Statistics distribution of relative hedge errors (in %)

statistic distribution of relative hedge errors	European crack spread option		Asian crack spread option	
	60-days	90-days	60-days	90-days
mean	7.1882	8.7246	11.7834	10.3991
standard error	0.0186	0.0194	0.0382	0.0519
minimum	0.0509	0.1188	0.0085	0.1080
maximum	22.8920	19.1026	53.6877	47.5615

Volatilities and correlations are the main parameters determining the price of a basket option,

⁶International Petroleum Exchange, acquired by the Intercontinental Exchange (ICE) in 2001.

and these cannot be directly observed. For valuation, we can use two types of correlations and volatilities: historical and implied. Implied volatilities and correlations, calculated from liquid option prices, reflect the expectation of market participants about these quantities over the remaining lifetime of the option. Historical volatilities and correlations tend to be stale since they are deduced from historical, i.e., past asset prices. Hence, using the implied volatilities and correlations would better reflect the current market. In practice, for pricing of basket and spread options, using the implied volatilities and the historical correlation is more realistic, because the implied volatilities can be obtained from prices of liquid options on individual assets, but implied correlations are much more difficult to obtain.

In our experiments, we estimate the historical correlations from the futures prices of the previous 60 or 90 trading days. This correlation is used to calculate the price of a spread option maturing in 60 or 90 trading days. In practice, one would use the implied volatilities of individual assets. Since we do not have complete historical data of option prices for the IPE Gasoil and Brent crude oil futures, we shall use instead (only for the purpose of this numerical study) the volatility based on the futures prices over the period of the option's life, the so-called realized volatility. In practice this is unrealistic (as it is a forward-looking algorithm), but we shall assume here that the implied volatility mimics the realized volatility (Christensen and Hansen (2002), Jiang and Tian (2005)). The option's hedging results are given in Table 3. The risk-free interest rate is taken 3% per annum.

For both 60- and 90-days crack spread options (European and Asian style), the means of the relative hedge errors are in the range of 7%-12%: rather small for commodity options. The average hedge errors obtained by using the combination of historical volatilities and historical correlation are much higher, and are in the range of 17%-21%, and can be as high as 50 % - 230 %. This is because, as Figure 8 shows, the historical volatility fails to represent the volatility during the option's life. In all, the application to real data in the oil markets shows that the GLN approach using the combination of realized volatilities and historical correlation performs well on the basis of delta hedging. Using the implied volatilities instead of the realized ones would likely lead to slightly worse, but comparable results, so the presented approach can be used as a "benchmark".

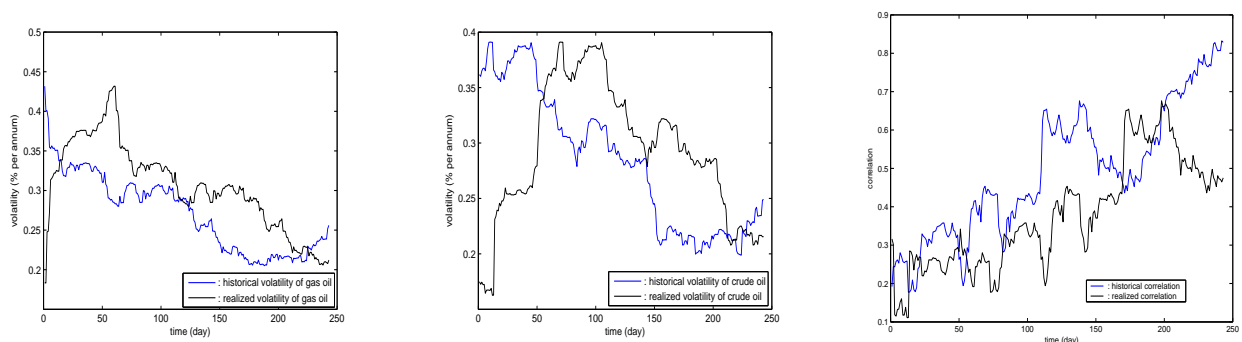


Figure 8: Historical and realized volatility of gasoil (*left*) and Brent crude oil (*middle*), and historical and realized correlation between gas oil and Brent crude oil (*right*) for 60 days window

Calculation of spread option's price involves the correlation between the underlying assets. If liquid spread option prices are available, it is possible to invert the spread option's price formula to

Table 4: ATM implied volatilities of NYMEX Brent crude oil and heating oil

commodity	trading day					
	12 Oct 2006	13 Oct 2006	16 Oct 2006	17 Oct 2006	18 Oct 2006	19 Oct 2006
Brent crude oil	31.26	32.16	30.98	31.42	31.07	30.65
heating oil	40.44	30.54	35.96	31.64	39.89	33.95

Table 5: Implied correlation between NYMEX Brent crude oil and heating oil (*italic indicates ATM option*)

strike price (\$/bbl)	trading day					
	12 Oct 2006	13 Oct 2006	16 Oct 2006	17 Oct 2006	18 Oct 2006	19 Oct 2006
X=15.0	0.1163	0.1991	0.4859	0.2585	0.3044	0.0517
X=14.5	0.3645	0.4421	0.6726	0.5001	0.5437	0.3466
X=14.0	0.5656	0.6405	0.8192	0.6911	0.7314	0.5827
X=13.5	<i>0.7318</i>	<i>0.7917</i>	<i>0.9322</i>	<i>0.8346</i>	<i>0.8774</i>	<i>0.7629</i>
X=12.0	<i>0.9765</i>	<i>0.9716</i>	*****	0.9688	*****	0.9575

obtain the implied correlation. This is very useful since the implied correlation should forecast the correlation between two underlying asset prices during the option's life much better than historical correlation. The GLN approach, being analytic, is particularly suited for implying the correlations from market prices of spread options. As most of the options in oil markets are Asian-style, our extension of the GLN method to Asian basket and spread options is particularly useful in this respect. For this application, we take the option price and the volatilities as given parameters (the ATM implied volatilities can be used for this purpose, obtained from liquid option prices on individual futures in the spread), and the correlation as the unknown one.

We estimate the implied correlation between NYMEX Brent crude oil and NYMEX heating oil futures, on the basis of the NYMEX crude oil/heating oil crack spread option prices (these are Asian-style), everything for delivery in December 2006. The daily option prices for the trading days October 12 - October 19, 2006 are used. We calculated first the implied volatilities of Brent crude oil and the heating oil by inverting the Black's formula, using corresponding option prices. However, these volatilities vary daily and across strikes. Figures 9 and 10 show the evolution of the implied volatility curve (vs. strike) for crude oil and heating oil. On the x -axis the time to maturity (in days) of the appropriate option is shown. If the ATM implied volatilities are not directly observable, we approximate them by the semi-parametric method introduced by Borovkova and Permana (2007).

The results are given in Table 4. Using these ATM volatilities and the spread option prices, we calculated the implied correlation using the GLN approach. The results are given in Table 5. The plots of the implied correlation vs. strike prices for six different trading days (12, 13, 16 - 19 October 2006) are shown in Figure 11. The implied correlations decrease for higher strike prices and are quite stable for different trading days. The graphs resemble the so-called *smirk*, or a *skew*, often observed in implied volatilities.

Inverting the GLN formula for the strike price 12 \$/bbl results in wrong implied correlations, higher than 1, because of mis-pricing option as a consequence of low liquidity. To estimate the

Table 6: Comparison between the call prices of 1:1 crude oil/heating oil crack spread option in the NYMEX and the prices calculated using the extrapolated implied correlations

call price	trading day					
	12 Oct 2006	13 Oct 2006	16 Oct 2006	17 Oct 2006	18 Oct 2006	19 Oct 2006
NYMEX call price (c)	1.33	1.11	0.96	0.99	1.06	1.05
estimated call price (\hat{c})	1.73	1.83	2.17	1.95	2.03	1.82

implied correlation from such options, we extrapolate the implied volatility curves as shown in Figure 11 and given in the last row of Table 5. Stars (*****) indicate that the extrapolation values are still higher than 1, since very close to 1. In such cases, we assume the implied correlation of such options are 1 (1.0459 and 1.0291). By those implied correlation values, we calculate the call prices of 1:1 crack spread option in the NYMEX given in Table 6. The obtained call prices are higher than the NYMEX call prices.

We would like to investigate the term structure of the implied correlation, i.e., its dependence on the option maturity month, but were unable to do so, due to the lack of liquid spread option prices for other maturities than December 2006.

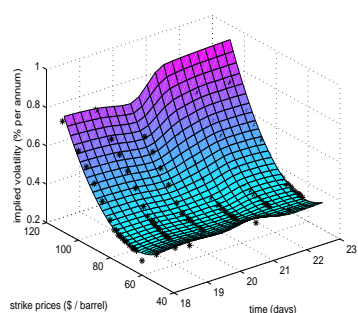


Figure 9: Evolution of NYMEX crude oil implied volatility curve (Oct. 12, 2006 - Oct. 19, 2006)

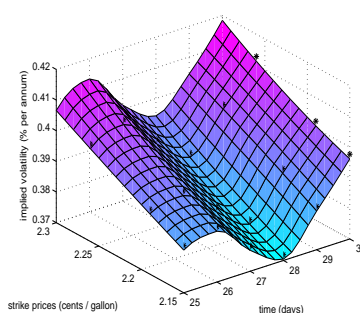


Figure 10: Evolution of NYMEX heating oil implied volatility curve (Oct. 12, 2006 - Oct. 19, 2006)

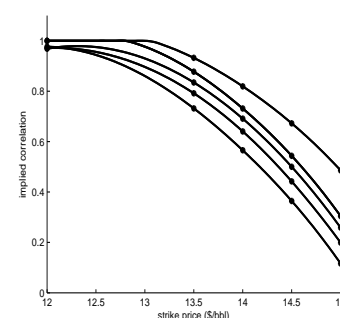


Figure 11: Implied correlation vs. strike price (Oct. 12 - 19, 2006)

5 Conclusions and future work

We introduced a new approach for pricing and hedging of Asian basket and spread options: derivatives widespread in energy markets. This approach uses a generalized family of lognormal distributions to approximate the distribution of the average basket value. The lognormal approximation allows us to use the Black-Scholes model, which leads to a closed-form solution for the Asian basket option price and the greeks. To our knowledge, this is the first semi-analytic method for

valuation and hedging of options on baskets and spreads. We demonstrated that our approach performs remarkably well in terms of option pricing and delta hedging, on the basis of both simulated and real market data. A closed form expression for the approximate spread option price allows us to calculate the implied correlation by inverting the option pricing formula.

There are several issues that deserve further investigation. Here we applied the method to spread options to imply the correlation between the underlying assets. For baskets consisting of more than two assets, the entire covariance matrix can be implied by the Marquadt optimization method, provided enough liquid basket option prices are available. This is feasible for equity or currency baskets, but for commodity baskets this is not currently feasible due to the low liquidity of traded options.

In this paper, we have used the ATM volatilities as inputs to the basket option pricing formula. This seems like a reasonable choice, given the special role that ATM volatility has. However, other choices of implied volatilities are possible, e.g. those at other strike prices⁷. Further investigation is necessary to ascertain whether the ATM volatilities are indeed the appropriate choice.

Here we have considered the most common type of commodity options: those whose underlyings are commodity futures contracts. If the method were applied to spread options on physical commodities (thus with commodity spot prices as underlyings), then the impact of the drift in the spot prices should be taken into consideration.

Finally, we assumed that the driving processes are the correlated geometric Brownian motions. Some commodity prices (such as electricity spot price) clearly contain jumps. The application of this approach to jump-diffusions is currently under investigation.

jumps

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⁷For example, many combinations of underlying commodity prices can lead to the at-the-money spread.

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Appendix: Formulae for deltas and vegas

Note: Everywhere, we use the following values of partial derivatives ($i = 1, \dots, N$):

$$\begin{aligned} \frac{\partial M_{2,i}}{\partial F_i} &= \frac{2}{n^2} \sum_{p_2=n_1}^{n_2} \sum_{p_1=n_1}^{n_2} F_i(0) \exp(\sigma_i^2 \min(t_{p_1}, t_{p_2})), \\ \frac{\partial \bar{M}_2}{\partial F_i} &= 2a_i \sum_{j=1}^N a_j F_j(0) \exp(\rho_{ij} \gamma_i \gamma_j) + 2a_i F_i(0) \frac{\partial \gamma_i}{\partial F_i} \sum_{j=1}^N \rho_{ij} \gamma_j a_j F_j(0) \exp(\rho_{ij} \gamma_i \gamma_j), \\ \frac{\partial \gamma_i}{\partial F_i} &= \frac{1}{2\gamma_i M_{1,i} M_{2,i}} \left(M_{1,i} \frac{\partial M_{2,i}}{\partial F_i} - 2M_{2,i} \right), \\ \frac{\partial \bar{M}_3}{\partial F_i} &= 3a_i \left(1 + F_i \frac{\partial \gamma_i}{\partial F_i} \right) (\rho_{ij} \gamma_j + \rho_{ik} \gamma_k) \sum_{k=1}^N \sum_{j=1}^N a_j a_k F_j(0) F_k(0) \exp(\rho_{ij} \gamma_i \gamma_j + \rho_{ik} \gamma_i \gamma_k + \rho_{jk} \gamma_j \gamma_k). \end{aligned}$$

The parameter V is given by the equations (17) or (18), according to the approximating distribution.

- Deltas using the shifted log-normal approximation:

$$\Delta_i = e^{-rT} \left[a_i \Phi(d_1) + (X - \tau) \phi(d_2) \frac{\partial V}{\partial F_i} + (\Phi(d_2) - \Phi(d_1)) \frac{\partial \tau}{\partial F_i} \right],$$

$$\begin{aligned} \partial V / \partial F_i &= \frac{1}{2(\bar{M}_1 - \tau)(\bar{M}_2 - 2\tau\bar{M}_1 + \tau^2)V} \left[(\bar{M}_1 - \tau) \frac{\partial \bar{M}_2}{\partial F_i} + \right. \\ &\quad \left. + 2a_i(\tau\bar{M}_1 - \bar{M}_2) + 2(\bar{M}_2 - \bar{M}_1^2) \frac{\partial \tau}{\partial F_i} \right]. \end{aligned}$$

- Deltas using the negative shifted log-normal approximation:

$$\Delta_i = e^{-rT} \left[(-X - \tau)\phi(-d_2) \frac{\partial V}{\partial F_i} + a_i \Phi(-d_1) + (\Phi(-d_1) - \Phi(-d_2)) \frac{\partial \tau}{\partial F_i} \right],$$

$$\begin{aligned} \partial V / \partial F_i &= \frac{1}{2(\bar{M}_1 + \tau)(\bar{M}_2 + 2\tau\bar{M}_1 + \tau^2)V} \left[(\bar{M}_1 + \tau) \frac{\partial \bar{M}_2}{\partial F_i} - \right. \\ &\quad \left. - 2a_i(\tau\bar{M}_1 + \bar{M}_2) - 2(\bar{M}_2 - \bar{M}_1^2) \frac{\partial \tau}{F_i} \right]. \end{aligned}$$

Note: everywhere we use the following values of partial derivatives:

$$\begin{aligned} \frac{\partial M_{2,i}}{\partial \sigma_i} &= \frac{2}{n^2} \sigma_i (F_i(0))^2 \sum_{p_2=n_1}^{n_2} \sum_{p_1=n_1}^{n_2} \min(t_{p_1}, t_{p_2}) \exp(\sigma_i^2 \min(t_{p_1}, t_{p_2})), \\ \frac{\partial \gamma_i}{\partial \sigma_i} &= \frac{1}{2\gamma_i M_{2,i}} \frac{\partial M_{2,i}}{\partial \sigma_i}, \\ \frac{\partial \bar{M}_2}{\partial \sigma_i} &= 2a_i F_i(0) \frac{\partial \gamma_i}{\partial \sigma_i} \sum_{j=1}^N \rho_{ij} \gamma_j a_j F_j(0) \exp(\rho_{ij} \gamma_i \gamma_j), \\ \frac{\partial \bar{M}_3}{\partial \sigma_i} &= 3a_i F_i(0) \frac{\partial \gamma_i}{\partial \sigma_i} \sum_{k=1}^N \sum_{j=1}^N a_j a_k F_j(0) F_k(0) (\rho_{ij} \gamma_j + \rho_{ik} \gamma_k) \exp(\rho_{ij} \gamma_i \gamma_j + \rho_{ik} \gamma_i \gamma_k + \rho_{jk} \gamma_j \gamma_k), \\ \frac{\partial \bar{M}_2}{\partial \rho_{ij}} &= 2a_i a_j F_i(0) F_j(0) \gamma_i \gamma_j \exp(\rho_{ij} \gamma_i \gamma_j), \\ \frac{\partial \bar{M}_3}{\partial \rho_{ij}} &= 6a_i a_j F_i(0) F_j(0) \gamma_i \gamma_j \sum_{k=1}^N a_k F_k(0) \exp(\rho_{ij} \gamma_i \gamma_j + \rho_{ik} \gamma_i \gamma_k + \rho_{jk} \gamma_j \gamma_k). \end{aligned}$$

- Vegas using the shifted log-normal approximation

$$\begin{aligned} \nu_{i,i} &= \partial c / \partial \sigma_i = e^{-rT} \left[(X - \tau)\phi(d_2) \frac{\partial V}{\partial \sigma_i} + \frac{\partial \tau}{\partial \sigma_i} (\Phi(d_2) - \Phi(d_1)) \right], \\ \nu_{i,j} &= \partial c / \partial \rho_{ij} = e^{-rT} \left[(X - \tau)\phi(d_2) \frac{\partial V}{\partial \rho_{ij}} + \frac{\partial \tau}{\partial \rho_{ij}} (\Phi(d_2) - \Phi(d_1)) \right], \quad i \neq j, \end{aligned}$$

$$\begin{aligned} \partial V / \partial \sigma_i &= \frac{1}{2(\bar{M}_1 - \tau)(\bar{M}_2 - 2\tau\bar{M}_1 + \tau^2)V} \left[(\bar{M}_1 - \tau) \frac{\partial \bar{M}_2}{\partial \sigma_i} + 2(\bar{M}_2 - \bar{M}_1^2) \frac{\partial \tau}{\partial \sigma_i} \right], \\ \partial V / \partial \rho_{ij} &= \frac{1}{2(\bar{M}_1 - \tau)(\bar{M}_2 - 2\tau\bar{M}_1 + \tau^2)V} \left[(\bar{M}_1 - \tau) \frac{\partial \bar{M}_2}{\partial \rho_{ij}} + 2(\bar{M}_2 - \bar{M}_1^2) \frac{\partial \tau}{\partial \rho_{ij}} \right]. \end{aligned}$$

- Vegas using the negative shifted log-normal approximation

$$\nu_{i,i} = \partial c / \partial \sigma_i = e^{-rT} \left[(-X - \tau) \phi(-d_2) \frac{\partial V}{\sigma_i} + \frac{\partial \tau}{\partial \sigma_i} (\Phi(-d_1) - \Phi(-d_2)) \right],$$

$$\nu_{i,j} = \partial c / \partial \rho_{ij} = e^{-rT} \left[(-X - \tau) \phi(-d_2) \frac{\partial V}{\rho_{i,j}} + \frac{\partial \tau}{\partial \rho_{ij}} (\Phi(-d_1) - \Phi(-d_2)) \right], \quad i \neq j,$$

$$\partial V / \partial \sigma_i = \frac{1}{2 (\bar{M}_1 + \tau) (\bar{M}_2 + 2\tau \bar{M}_1 + \tau^2) V} \left[(\bar{M}_1 + \tau) \frac{\partial \bar{M}_2}{\partial \sigma_i} - 2(\bar{M}_2 - \bar{M}_1^2) \frac{\partial \tau}{\partial \sigma_i} \right],$$

$$\partial V / \partial \rho_{ij} = \frac{1}{2 (\bar{M}_1 + \tau) (\bar{M}_2 + 2\tau \bar{M}_1 + \tau^2) V} \left[(\bar{M}_1 + \tau) \frac{\partial \bar{M}_2}{\partial \rho_{ij}} - 2(\bar{M}_2 - \bar{M}_1^2) \frac{\partial \tau}{\partial \rho_{ij}} \right].$$